

Remarks on a Liouville-type theorem for Beltrami flows

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Abstract

We present a simple, short and elementary proof that if v is a Beltrami flow with a finite energy in \mathbb{R}^3 then $v = 0$. In the case of the Beltrami flows satisfying $v \in L_{loc}^\infty(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ with $q \in [2, 3)$, or $|v(x)| = O(1/|x|^{1+\varepsilon})$ for some $\varepsilon > 0$, we provide a different, simple proof that $v = 0$.

AMS Subject Classification Number: 35Q31, 76B03, 76W05

keywords: Euler equations, Beltrami flows, Liouville type theorem

1 Introduction

Ideal homogeneous incompressible inviscid fluid flows are governed by the Euler equations:

$$(E) \begin{cases} \frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\ \operatorname{div} v = 0, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \end{cases}$$

where $v = (v_1, \dots, v_n)$, $v_j = v_j(x, t)$, $j = 1, \dots, n$, $n \geq 2$, is the velocity of the flow, $p = p(x, t)$ is the scalar pressure. Let R_j , $j = 1, \dots, n$, denote the Riesz transforms, given by

$$R_j(f)(x) = C_n \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{(x_j - y_j)f(y)}{|x - y|^{n+1}} dy, \quad C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}.$$

The pressure in the Euler (and Navier-Stokes) equations is given in terms of the velocity up to addition of a harmonic function by

$$p = \sum_{j,k=1}^n R_j R_k (v_j v_k). \quad (1.1)$$

This is easily seen by taking the divergence of (E). In [2](see also [1]) the following result is obtained.

Theorem 1.1 *If (v, p) satisfies (1.1) and $|p| + |v|^2 \in L^1(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} v_j v_k dx = -\delta_{jk} \int_{\mathbb{R}^n} p dx. \quad (1.2)$$

In the next section we present a simple proof of this result using the continuity of the Fourier transform of functions belonging to $L^1(\mathbb{R}^n)$. In order to see the implications of the above theorem for Beltrami flows, let us recall that in the stationary case in \mathbb{R}^3 , the first equations of (E) can be rewritten as

$$v \times \omega = \nabla(p + \frac{1}{2}|v|^2), \quad \omega = \operatorname{curl} v. \quad (1.3)$$

A vector field v is called a Beltrami flow if there exists a function $\lambda = \lambda(x)$ such that

$$\omega = \lambda v. \quad (1.4)$$

Therefore, if v is a Beltrami flow, then the pair (v, p) is a solution of the stationary Euler equations if

$$p + \frac{1}{2}|v|^2 = c, \quad c = \text{constant}. \quad (1.5)$$

We call such a solution (v, p) a “Beltrami solution” of the stationary Euler equations. We refer to [3] for a recent interesting result regarding the Beltrami flows. Recently, Nadirashvili proved a Liouville type property of Beltrami flows ([4]). He showed that a Beltrami solution (v, p) satisfying either $v \in L^q(\mathbb{R}^3)$, $2 \leq q \leq 3$, or $|v(x)| = o(1/|x|)$ is necessarily trivial, $v = 0$. In the case of finite energy Beltrami flows we have the following immediate consequence of Theorem 1.1:

Theorem 1.2 *Let (v, p) be a Beltrami solution of the stationary Euler equations with the λ given in (1.4). If $v \in L^2(\mathbb{R}^3)$, then $v = 0$. The same conclusion holds, for instance, if there exists $q \in [\frac{6}{5}, \infty]$ such that $v \in L^q(\mathbb{R}^3)$ and $\lambda \in L^{\frac{6q}{5q-6}}(\mathbb{R}^3)$ (if $v \in L^{\frac{6}{5}}(\mathbb{R}^3)$, then we require $\lambda \in L^\infty(\mathbb{R}^3)$).*

We have also the following result for the cases considered in the paper [4], for which we present a different, simple proof.

Theorem 1.3 *Let $v \in L_{loc}^\infty(\mathbb{R}^3)$ be a Beltrami solution of the stationary Euler equations satisfying either $v \in L^q(\mathbb{R}^3)$ for some $q \in [2, 3)$, or that there exists $\varepsilon > 0$ such that $|v(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \rightarrow \infty$. Then, $v = 0$.*

2 Proof of the Theorems

We use the notation for the Fourier transform of $f(x)$

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx,$$

whenever the right hand side is defined. In terms of the Fourier transform the Riesz transform is defined as

$$\widehat{R_j(f)}(\xi) = \frac{i\xi_j}{|\xi|}, \quad i = \sqrt{-1}.$$

Proof of Theorem 1.1 Without loss of generality we may restrict ourselves to the stationary case, $v(x, t) = v(x)$, $p(x, t) = p(x)$. By the Fourier transform one has

$$\hat{p}(\xi) = - \sum_{j,k=1}^n \frac{\xi_j \xi_k}{|\xi|^2} \widehat{v_j v_k}(\xi). \quad (2.1)$$

We note that $\hat{p}(\xi)$ and $\widehat{v_j v_k}(\xi)$, $j, k = 1, \dots, n$, are continuous at $\xi = 0$ from the hypothesis, $|p| + |v|^2 \in L^1(\mathbb{R}^n)$. Let w be a given constant vector with $|w| = 1$. We put $\xi = \rho w$ in (2.1), and pass $\rho \rightarrow 0$ to obtain

$$\int_{\mathbb{R}^n} p dx = - \int_{\mathbb{R}^n} (v \cdot w)^2 dx. \quad (2.2)$$

If we plug $w = \mathbf{e}^j$ in (2.2), where \mathbf{e}^j is the canonical basis of \mathbb{R}^n with its components given by $(\mathbf{e}^j)_k = \delta_{jk}$, then we have

$$\int_{\mathbb{R}^n} p dx = - \int_{\mathbb{R}^n} v_j^2 dx \quad \forall j = 1, \dots, n.$$

On the other hand, for $j \neq k$, if we put $w = \frac{\mathbf{e}^j + \mathbf{e}^k}{\sqrt{2}}$ in (2.2), we obtain $\int_{\mathbb{R}^n} v_j v_k dx = 0$. \square

Proof of Theorem 1.2 Since (v, p) is a Beltrami solution of stationary Euler equations, we have $p - c = \frac{1}{2}|v|^2 := \tilde{p}$ for some constant c . In the case $v \in L^2(\mathbb{R}^3)$ we find that $|v|^2 + |\tilde{p}| \in L^1(\mathbb{R}^3)$, and by Theorem 1.1 we obtain

$$\int_{\mathbb{R}^3} \tilde{p} dx = -\frac{1}{3} \int_{\mathbb{R}^3} |v|^2 dx = -\frac{1}{2} \int_{\mathbb{R}^3} |v|^2 dx,$$

which implies that $v = 0$, and $\tilde{p} = \sum_{j,k=1}^n R_j R_k (v_j v_k) = 0$. On the other hand, if $v \in L^q(\mathbb{R}^3)$ and $\lambda \in L^{\frac{6q}{5q-6}}(\mathbb{R}^3)$ with $\frac{6}{5} < q \leq \infty$, or $v \in L^{\frac{6}{5}}(\mathbb{R}^3)$ and $\lambda \in L^\infty(\mathbb{R}^3)$, then we estimate

$$\begin{aligned} \|v\|_{L^2} &\leq C \|\nabla v\|_{L^{\frac{6}{5}}} \leq C \|\omega\|_{L^{\frac{6}{5}}} = C \|\lambda v\|_{L^{\frac{6}{5}}} \\ &\leq C \|\lambda\|_{L^{\frac{6q}{5q-6}}} \|v\|_{L^q} < \infty, \end{aligned}$$

and we reduce to the above case of $v \in L^2(\mathbb{R}^3)$. \square

Proof of Theorem 1.3 We first observe that our hypothesis implies that

$$\int_{\mathbb{R}^3} |v|^2 |x|^{\lambda-2} dx < \infty. \quad (2.3)$$

for some $\lambda \in (1, 2)$. Indeed, the case $|v(x)| = O(1/|x|^{1+\varepsilon})$ as $|x| \rightarrow \infty$ is obvious, while in the case $v \in L^q(\mathbb{R}^3)$ for some $q \in [2, 3)$, we have the following estimate,

$$\int_{\{|x| \geq 1\}} |v|^2 |x|^{\lambda-2} dx \leq C \|v\|_{L^q}^2 \left(\int_1^\infty r^{2-\frac{q(\lambda-2)}{q-2}} dr \right)^{\frac{q-2}{q}} < \infty$$

for λ with $1 < \lambda < \frac{6}{q} - 1$. Let us introduce a standard radial cut-off function $\sigma \in C_0^\infty(\mathbb{R}^N)$ such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases} \quad (2.4)$$

and $0 \leq \sigma(x) \leq 1$ for $1 < |x| < 2$. Then, for each $R > 0$, we define $\sigma\left(\frac{|x|}{R}\right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N)$. A Beltrami solution (v, p) with $p = -\frac{1}{2}|v|^2 + C$ satisfies

$$\sum_{j,k=1}^3 \int_{\mathbb{R}^3} v_j v_k \partial_j \partial_k \varphi dx = \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \Delta \varphi dx \quad \forall \varphi \in C_0^2(\mathbb{R}^3). \quad (2.5)$$

We choose our test function $\varphi(x) = \varphi_{\delta,R}(x) = (|x|^{2\lambda} + \delta)^{\frac{1}{2}} \sigma_R$ for $\delta, R > 0$ in (2.5), which is an approximation of $\varphi = |x|^\lambda$, and passing first $\delta \rightarrow 0$, and then $R \rightarrow \infty$, using the dominated convergence theorem, taking (2.3) into account, we obtain easily that

$$(\lambda - 1) \int_{\mathbb{R}^3} |v|^2 |x|^{\lambda-2} dx = 2(\lambda - 2) \int_{\mathbb{R}^3} (v \cdot x)^2 |x|^{\lambda-4} dx. \quad (2.6)$$

The fact that $\lambda \in (1, 2)$ in (2.6) implies $v = 0$. \square

Acknowledgements

DC was partially supported by NRF grants 2006-0093854 and 2009-0083521. PC was partially supported by NSF DMS grants 1209394 and 1265132.

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